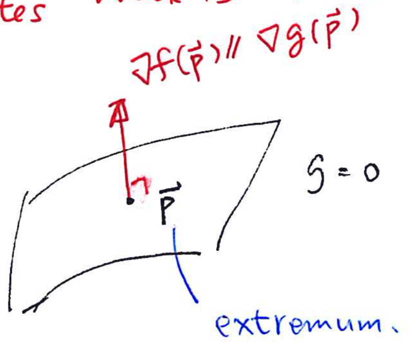
MATH 2010B Advanced Calculus I Lecture Notes Week 13 (Martin Li)



<u>Last time</u> ... "Lagrange Multiplier".

$$\begin{cases} \text{max/min} & f(x,y,z) \\ \text{under} & g(x,y,z) = 0 \end{cases}$$

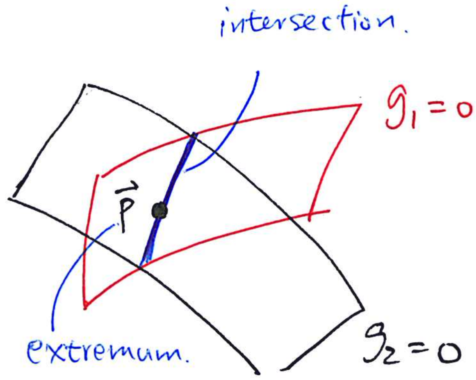
At an extremum $\vec{p}$,

$$\begin{cases} \nabla f(\vec{p}) = \lambda \nabla g(\vec{p}) \\ g(\vec{p}) = 0. \end{cases}$$

[<u>Non-deg. condition</u>

$\nabla g(\vec{p}) \neq \vec{0}$.]

Optimization with multiple constraints



<u>Problem</u>:

$$\begin{cases} \text{max/min} & f(x,y,z) \\ \text{under} & g_1(x,y,z) = 0 \\ & g_2(x,y,z) = 0 \end{cases}$$

<u>Theorem</u>: At extremum $\vec{p}$, then

5 unknowns

5 eq$^{\underline{n}}$

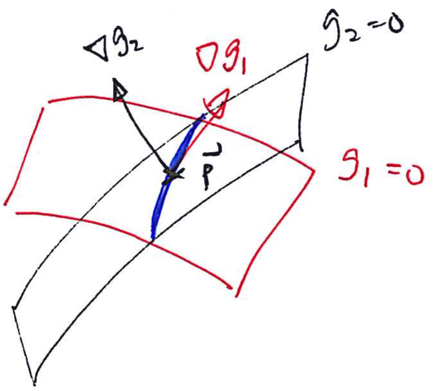
$$\begin{cases} \nabla f(\vec{p}) = \lambda_1 \nabla g_1(\vec{p}) + \lambda_2 \nabla g_2(\vec{p}) \\ g_1(\vec{p}) = 0 \\ g_2(\vec{p}) = 0. \end{cases}$$

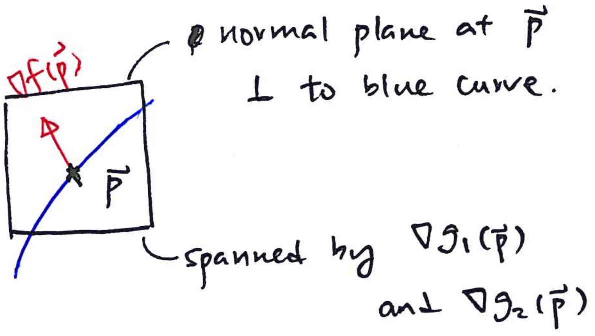
Lagrange multipliers

<u>Idea</u>: # multipliers = # constraints

<u>Non-deg. condition</u>: $\nabla g_1(\vec{p})$ & $\nabla g_2(\vec{p})$ are "linearly independent"

(i.e. they are not parallel)





E.g. 1 : $\begin{cases} \max & f(x,y,z) = x^2+2y-z^2 \\ \text{under} & g_1(x,y,z) = 2x-y=0 \\ & g_2(x,y,z) = y+z=0 \end{cases}$

Use Lagrange multiplier,

$$\left.\begin{cases} \nabla f = (2x, 2, -2z) \\ \nabla g_1 = (2, -1, 0) \\ \nabla g_2 = (0, 1, 1) \end{cases}\right. \quad \text{not parallel} \Rightarrow \text{non-deg. o.k!}$$

the system:

$$\begin{cases} 2x = 2\lambda_1 & \text{—— ①} \\ 2 = -\lambda_1 + \lambda_2 & \text{—— ②} \\ -2z = \lambda_2 & \text{—— ③} \\ 2x - y = 0 & \text{—— ④} \\ y + z = 0 & \text{—— ⑤} \end{cases}$$

② $\Rightarrow$ $\lambda_1 = \lambda_2 - 2$. —— ⑥

④ + ⑤ $\Rightarrow$ $2x = y = -z$ $\qquad \Rightarrow \lambda_1 = \frac{2}{3},\ \lambda_2 = \frac{8}{3}.$

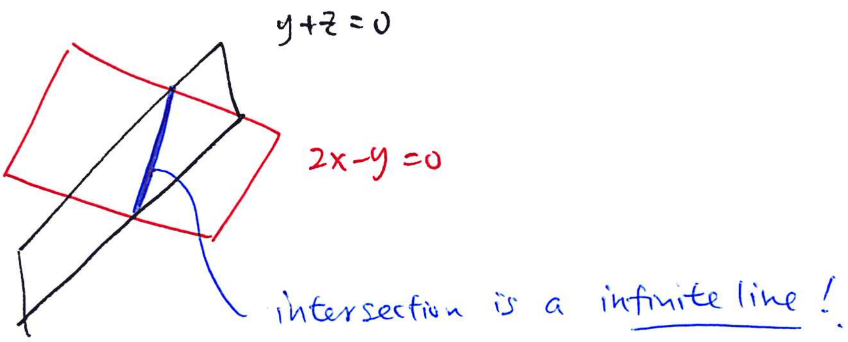
① + ③ $\Rightarrow$ $2\lambda_1 = \frac{1}{2}\lambda_2$ —— ⑦

Putting back $\Longrightarrow$ $x = \frac{2}{3}$, $y = \frac{4}{3}$, $z = -\frac{4}{3}$.

Ex?

$\Longrightarrow$ get extremum (actually a max) at ↑ s.t.

$$f\left(\tfrac{2}{3}, \tfrac{4}{3}, -\tfrac{4}{3}\right) = \tfrac{4}{3}$$ #



Application:

Show the AM/GM inequality: $a, b, c \geq 0$

$$(abc)^{\frac{1}{3}} \leq \frac{a+b+c}{3}$$

GM: geometric mean

AM: arithemetic mean

Proof using optimization:

$$(\star) \begin{cases} \max & f(x,y,z) = x^2y^2z^2 \\ \text{under} & \underbrace{x^2+y^2+z^2}_{g(x,y,z)} = r^2 \end{cases} \qquad 0<r: \text{fixed}.$$

[Take $a = x^2$, $b = y^2$, $c = z^2 \geq 0$.]

· To solve $(\star)$, $\nabla f = (2xy^2z^2, 2x^2yz^2, 2x^2y^2z)$

$\nabla g = (2x, 2y, 2z)$

$$\begin{cases} 2xy^2z^2 = 2\lambda x \\ 2x^2yz^2 = 2\lambda y \\ 2x^2y^2z = 2\lambda z \\ x^2+y^2+z^2 = r^2. \end{cases}$$

Case 1: $\lambda = 0 \Rightarrow x$ or y or $z = 0$.

$f = x^2y^2z^2 \geq 0$ but $f = 0$ so not max.

Case 2: $\lambda \neq 0$, $x, y, z \neq 0$.

Cancel $\Rightarrow \begin{cases} y^2z^2 = \lambda \\ x^2z^2 = \lambda \\ x^2y^2 = \lambda. \end{cases} \Rightarrow x^2 = y^2 = z^2.$

put back

$\Downarrow$

Sol: $x^2 = y^2 = z^2 = \frac{r^2}{3}$.

at these pts, $f = \left(\frac{r^2}{3}\right)^3$.

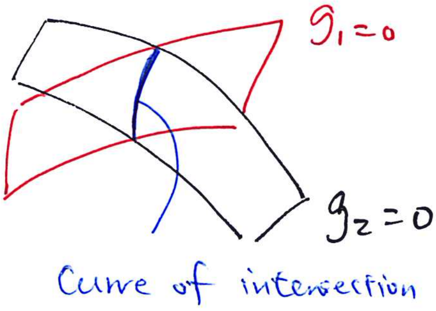
$$\Rightarrow \quad abc \leq \left(\frac{a+b+c}{3}\right)^3 \quad \xRightarrow{\text{cube root}} \quad \text{AM-GM inequality.}$$

General AM-GM: $(a_1a_2\cdots a_n)^{\frac{1}{n}} \leq \frac{a_1+a_2+\cdots+a_n}{n} \quad \forall a_i \geq 0.$

Ex: Prove this.

Last time ... Optimization with multiple constraints

$$\begin{cases} \max/\min & f(x,y,z) \\ \text{under} & g_1(x,y,z) = 0 \\ & g_2(x,y,z) = 0 \end{cases}$$



Lagrange multipliers: λ_1 & λ_2

$$\begin{cases} \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \\ g_1 = 0 \\ g_2 = 0 \end{cases}$$

3

2

5 unknowns: $x, y, z, \lambda_1, \lambda_2$

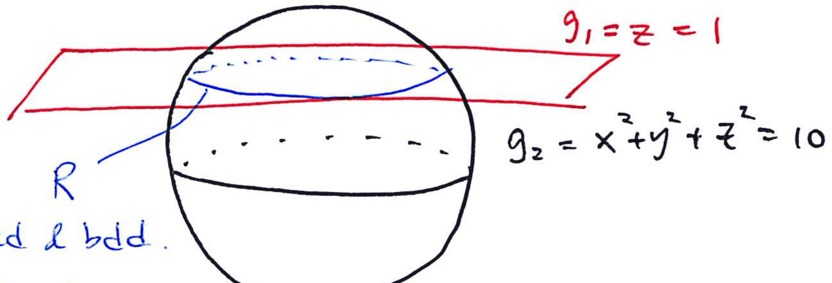
5 equations

Example:
$$\begin{cases} \max/\min & f(x,y,z) = x^2yz + 1 \\ \text{under} & z = 1 \\ & x^2+y^2+z^2 = 10 \end{cases}$$

Recall: $f: R \longrightarrow \mathbb{R}$ cts, $R \subset \mathbb{R}^3$

$\Rightarrow$ If R is compact, i.e. closed & bounded then $\max f$ & $\min f$ exists and they are achieved in R

Q: What is R in this case?



closed & bdd.
and f is cts
$\Downarrow$
max & min exist.

Solution 1: (Lagrange multiplier w/. 2 constraints)

$$\nabla f = \begin{pmatrix} 2xyz \\ x^2 z \\ x^2 y \end{pmatrix} ; \quad \nabla g_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} ; \quad \nabla g_2 = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}.$$

$$\begin{cases} \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \\ g_1 = 1 \\ g_2 = 10 \end{cases} \Rightarrow \begin{cases} 2xyz = \lambda_2 (2x) \\ x^2 z = \lambda_2 (2y) \\ x^2 y = \lambda_1 + \lambda_2 (2z) \\ \boxed{z = 1} \\ x^2 + y^2 + z^2 = 10 \end{cases}.$$

Sub $z=1$ into others:

$$\begin{cases} 2xy = 2x\lambda_2 & \text{--- (1)} \\ x^2 = 2y\lambda_2 & \text{--- (2)} \\ x^2 y = \lambda_1 + 2\lambda_2 & \text{--- (3)} \\ x^2 + y^2 = 9 & \text{--- (4)} \end{cases}$$

Case 1: $\boxed{x = 0}$

(4) $\Rightarrow \boxed{y = \pm 3} \Rightarrow$ 2 solutions: $(0, \pm 3, 1)$

Case 2: $x \neq 0$

(1) $\Rightarrow y = \lambda_2$.

Sub into (2) & (3)

$$\begin{cases} x^2 = 2y^2 & \text{--- (5)} \\ x^2 y = \lambda_1 + 2y. & \text{--- (6)} \\ x^2 + y^2 = 9 & \text{--- (4)} \end{cases}$$

(4) & (5) $\Rightarrow 2y^2 + y^2 = 9$

$\Rightarrow y^2 = 3 \Rightarrow y = \pm\sqrt{3}$.

and $x^2 = 2y^2 = 6 \Rightarrow x = \pm\sqrt{6}$.

$\Rightarrow$ 4 solutions: $(\pm\sqrt{6}, \pm\sqrt{3}, 1)$

Check which ones are max/min: $f(x,y,z) = x^2 yz + 1$

$f(0, \pm 3, 1) = 1$

$f(\pm\sqrt{6}, \pm\sqrt{3}, 1) = \pm 6\sqrt{3} + 1$

$\Rightarrow$ max $= 1 + 6\sqrt{3}$ at $(\pm\sqrt{6}, \sqrt{3}, 1)$

min $= 1 - 6\sqrt{3}$ at $(\pm\sqrt{6}, -\sqrt{3}, 1)$

Non-deg condition: $\nabla g_1 \not\parallel \nabla g_2$ (Check!)

Solution 2: (reducing the number of constraints)

2 constraints: $\begin{cases} z = 1 \\ x^2+y^2+z^2 = 10 \end{cases} \Rightarrow \begin{cases} x^2+y^2 = 9 \\ z = 1 \end{cases}$

Problem is the same as: (put in $z=1$)

$$(\text{A}) \begin{cases} f(x,y) = x^2y+1 \\ \text{under } x^2+y^2 = 9. \end{cases}$$

2-variables.
1-constraint.
$\Downarrow$
Lagrange multiplier λ
(Ex: Do it this way!)

In this case, use polar coordinates:

$$\boxed{x = 3\cos\theta,\ y = 3\sin\theta}$$

(A) reduces to $\begin{cases} f(\theta) = 27\cos^2\theta\sin\theta+1 \\ \text{no constraint on } \theta. \end{cases}$

Set $f'(\theta) = 0 \Rightarrow -2\cos\theta\underbrace{\sin^2\theta}_{1-\cos^2\theta}+\cos^3\theta = 0$

$\cos\theta = 0$ or $-2(1-\cos^2\theta)+\cos^2\theta = 0$

$\cos^2\theta = \frac{2}{3}.$

$\cos\theta = \pm\sqrt{\frac{2}{3}}.$

① $\cos\theta = 0 \Rightarrow x = 0,\ y = \pm 3$

② $\cos\theta = \pm\sqrt{\frac{2}{3}} \Rightarrow x = \pm\sqrt{6},\ y = \pm\sqrt{3}.$

Exam covers up to here ....

Two important theorems in Differential Calculus

Problem 1: Given $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is C^1 function,

for what values of $y \in \mathbb{R}^n$ can we solve

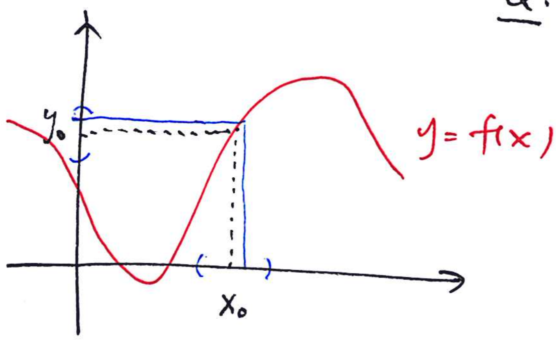
$$f(x) = y \quad \text{for } x.$$

in other words, when does the inverse

$$f^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n \text{ exist ?}$$

Ans: "Locally", we can answer that using derivatives of f.

basically: $(n=1)$ $\quad f : \mathbb{R} \longrightarrow \mathbb{R}$



Q: Can we solve $f(x) = y$ locally near (x_0, y_0) ?

i.e. Can we find some $\varepsilon > 0$ s.t.

$\forall\ y \in (y_0 - \varepsilon, y_0 + \varepsilon)$

$\Rightarrow \exists$ unique $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$

s.t. $\quad f(x) = y$.

Ans: $f'(x_0) \neq 0 \Rightarrow$ We can always solve $f(x) = y$ locally near x_0.

Higher Dimensional Case:

(**) **Inverse Function Theorem**:

Given $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a C^1 function

and $f(x_0) = y_0$

Assume $Df(x_0)$ is non-singular, ie $\boxed{\det(Df(x_0)) \neq 0}$.

$n \times n$ matrix

Then, $\forall\ y \approx y_0$, we can find a unique $x \approx x_0$ s.t

$$f(x) = y$$

Problem 2: Given $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f = f(x,y)$, C^1 function,

consider the "level curve": $f(x,y) = 0$

does this equation defines an implicit function $y = y(x)$?

ie $f(x, y(x)) \equiv 0$. $\forall x$.

Example: consider the equation

$$x^2 + y^2 = 1$$

$\Rightarrow$ Solve y in terms of x:

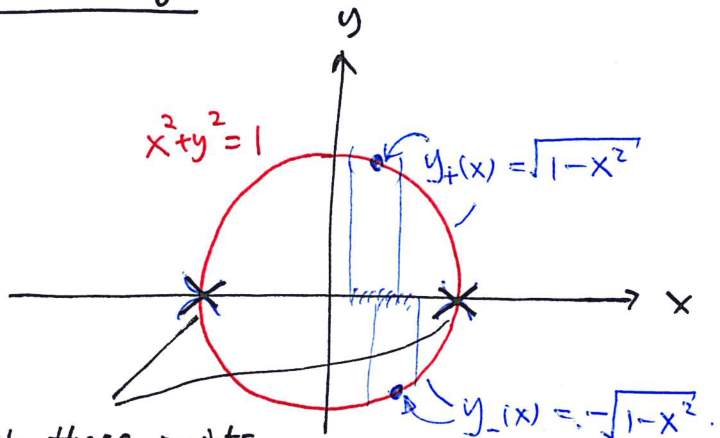
$$y = \pm\sqrt{1-x^2}$$

Remarks:

① no uniqueness. $y_+(x)$, $y_-(x)$

② $y(x)$ is not diff. at $x = \pm 1$

Geometrically:



at these points, the curve is not a graph of any function in x but it is O.K. to write it as a function y.

Get implicit function $y = y(x)$

$\Updownarrow$

$f(x,y) = 1$ is the graph of the function $y(x)$.

Q: At (x_0, y_0) on a level curve $f(x,y) = c$, then can we express the curve locally as the graph of a function in x (or y) ?

Answer is :

Implicit function theorem

Let $f(x,y) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a C^1 function.

and (x_0, y_0) is a point on the level curve

$$C := \{ f(x,y) = c \} \qquad c = \text{constant}.$$

If $\boxed{\dfrac{\partial f}{\partial y}(x_0, y_0) \neq 0}$, then we can express C

locally near (x_0, y_0) as the graph of a function

$$y = g(x) \quad \text{i.e.} \quad f(x, g(x)) \equiv c$$

locally defined near x_0.

Why? Back to the example $f(x,y) = x^2 + y^2 = 1$

$$\frac{\partial f}{\partial y} = 2y = 0 \iff y = 0 \ \& \ x = \pm 1.$$

Why is there a problem at $x = \pm 1$ $(y = 0)$.

Recall: Implicit diff.

$$f(x, y(x)) = 1$$

diff. in $x \Rightarrow$ $$f_x + f_y \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}}$$

becomes ∞ if $f_y = 0$ but $f_x \neq 0$.

Ex: What if we want to express it as a graph of a function in y?

Ans: $\dfrac{\partial f}{\partial x}(x_0, y_0) \neq 0$.

If both $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$ at (x_0, y_0), then we cannot say anything!